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Thermodynamic properties of the Dicke model in the strong-coupling regime

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Abstract. We discuss the problem of a N two-level systems interacting with a single radiation mode in the strong-coupling regime. The thermodynamic properties of Dicke model are analyzed developing a perturbative expansion of the partition function in the high-temperature limit and we use this method to investigate the connections between the Dicke and the collective one-dimensional Ising model.

1 Introduction

The thermodynamic properties of the Dicke model [1] for N two-level system interacting with a single mode of the radiation field have been studied extensively by different authors and with different methods [2–9]. Interest in this model has been renewed by a number of theoretical works, where it is discussed in connection with quantum chaos [10,11] and entanglement [12–14] and for various physical systems as photonic band gap materials [15,16] and Josephson junctions [17].

The Dicke model exhibits a second-order phase transition and, after the first derivation due to Hepp and Lieb [2], a simple computational method was provided by Wang and Hioe [3] based on the use of Glauber's coherent states [18] for the radiation field and on the assumption that, in the thermodynamic limit ($N \to \infty$, $V \to \infty$ but $\rho = N/V$ finite), the field operators can be treated as c-number functions. The Wang and Hioe method has been recently applied by Lee and Johnson [16,19] to derive the thermodynamic properties for an extended version of the Dicke Hamiltonian incorporating spin-spin and spin-boson interactions. Perturbative methods have recently been proposed to perform thermodynamical calculations without to have recourse to the Wang and Hioe computational method. In reference [20] it has been developed a perturbative expansion of partition function and a simple analytic solution is found for high coupling constant. The existence of this soluble model allows to show that the interaction of independent atomic spins with a resonant photon mode can be interpreted as an effective

spin-spin interaction of long range nature. The similarity between the Dicke Model and the collective XY-model has been found in reference [14], using another perturbative approach in order to derive a temperature-dependent effective atomic Hamiltonian. The prediction of a phase transition has been criticized as obtained for nonrealistic Hamiltonian, the simple Dicke Hamiltonian in which both the counter rotating terms and the diamagnetic term are truncated, that violated gauge invariance. As a matter of fact, the critical properties do not changes qualitatively when the counter rotating terms are taken into account [5] but, when the diamagnetic term is included, the Super-radiant Phase Transition (SPT) is forbidden to occur [21]. The same conclusion has been obtained in reference [20] for a model Hamiltonian incorporating the effects of the diamagnetic term into a new frequency of the photon mode [22].

In this paper we present a formal analysis which generalizes the results of reference [20] beyond the resonant condition and gives an alternative derivation of some already known and recent results on the subject. In the next section we analyze the procedure that permit to develop a perturbative expansion of the partition function as generally found in literature. The Dicke Model is introduced in Section 3 and a simple expression for the partition function is analytically derived in Sections 4 and 5 for high value of the coupling constant. This general result, as shown in Appendix A, is independent of the choice for the basis for expressing the state of radiation field. Some special cases of interest are explored and the salient features of the thermodynamic phase transition are discussed in Section 6 and in Section 7, where we also discuss the analogies between the Dicke model and the collective

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one-dimensional Ising model. In Section 8 we draw our final conclusions.

2 Perturbation expansion of partition function

The starting point for deriving the thermodynamic properties of a quantum model defined by an Hamiltonian H is the partition function:

$$Z(N,T) = \text{Tr}\left\{e^{-\beta H}\right\} \tag{1}$$

It is generally difficult to diagonalize $e^{-\beta H}$ and a variety of systematic approximations has been proposed [23,24,27]. A convenient procedure is to introduce a separation

$$H = H_0 + H_I \tag{2}$$

such that the exponential operator of (1) can be disentangled into a product of an infinite series of exponential operators as

$$e^{-\beta(H_0+H_I)} = e^{-\beta H_0} e^{-\beta H_I} \prod_{i=2}^{\infty} e^{(-\beta)^i C_i} \tag{3}$$

where C_i is a homogeneous polynomial of degree n in H_0 and H_I. All the $C_i{'}$s contain the commutator $[H_I, H_0]$ and using the method given by Wilcox [26] they can be determined as:

$$C_2 = \frac{1}{2}[H_I, H_0] \tag{4}$$

$$C_3 = -\frac{1}{6}[H_0, [H_I, H_0]] - \frac{1}{3}[H_I, [H_I, H_0]] \tag{5}$$

with increasing complexity for higher i. The basic step in the construction of approximants to the equation (3) is to find a product of exponential operators which is correct up to a certain power of β. The disentangled and undisentangled form of equation (3) are expanded in terms of β and operator coefficients of equal power of β are compared. The result is that

$$e^{-\beta(H_0+H_I)} = e^{-\beta H_0} e^{-\beta H_I} + O(\beta^2). \tag{6}$$

A more accurate approximation is obtained by introducing the following symmetrized approximation of the Hermitian operator $e^{-\beta H}$:

$$e^{-\beta(H_0+H_I)} = e^{-\beta H_0/2} e^{-\beta H_I} e^{-\beta H_0/2} + O(\beta^3), \tag{7}$$

Obviously, the partition function obtained with using (7) and (6) is identical due to the cyclic permutation property of the trace. The error induced by the approximation

$$\text{Tr}\left\{e^{-\beta(H_0+H_I)}\right\} = \text{Tr}\left\{e^{-\beta H_0} e^{-\beta H_I}\right\} \tag{8}$$

can be estimated using the Hermitian approximation

$$e^{-\beta(H_0+H_I)} =$$
$$e^{-\beta H_0/2} e^{-\beta H_I/2} e^{-\beta^3 C_3/4} e^{-\beta H_I/2} e^{-\beta H_0/2} + O(\beta^5) \tag{9}$$

and one obtains [24]

$$\left|\text{Tr}\left\{e^{-\beta(H_0+H_I)} - e^{-\beta H_0/2} e^{-\beta H_I} e^{-\beta H_0/2}\right\}\right|$$
$$< \frac{\beta^3}{4}\left|\text{Tr}\left\{C_3\right\}\right|\left|\text{Tr}\left\{e^{-\beta H_0} e^{-\beta H_I}\right\}\right|. \tag{10}$$

3 The model

We use this method to study the thermodynamic properties of the Dicke model Hamiltonian ($\hbar = c = 1$)

$$H = \omega a^\dagger a + \sum_{i=1}^{N}\left[\frac{\epsilon}{2}\sigma_i^z + \frac{\lambda}{\sqrt{N}}(a^\dagger + a)(\sigma_i^+ + \sigma_i^-)\right]. \tag{11}$$

Here, ω is the frequency of a single mode of radiation, ϵ is the energy difference between the two levels of N identical spin-$\frac{1}{2}$ systems, σ_i^z, σ_i^+ and σ_i^- are respectively the z component, the raising and the lowering operators of the Pauli matrices used to describe the ith spin, a and $a^\dagger$ are the annihilation and creation operators for photons. For N two-level atoms the coupling constant is

$$\lambda = \epsilon d\sqrt{\frac{2\pi\rho}{\omega}} \tag{12}$$

where d is the projection of the transition dipole moment on the polarization vector of the field mode and ρ the density of the atoms.

We make the following separation:

$$H_0 = \omega a^\dagger a, \tag{13}$$

$$H_I = \frac{\epsilon}{2}S^z + \frac{\lambda}{\sqrt{N}}(a^\dagger + a)(S^+ + S^-) \tag{14}$$

where

$$S^{(z,\pm)} = \sum_{i=1}^{N}\sigma_i^{(z,\pm)} \tag{15}$$

are the collective atomic operators. By applying the harmonic-oscillator commutation relations of the field mode operators $a^\dagger$, a:

$$[a, a^\dagger] = 1, \quad [a^\dagger a, a^\dagger] = a^\dagger, \quad [a^\dagger a, a] = -a \tag{16}$$

and that for the atomic operators:

$$[S^+, S^-] = S^z, \quad [S^z, S^\pm] = \pm 2S^\pm \tag{17}$$

it is easy to show that

$$C_2 = \frac{\lambda\omega}{2\sqrt{N}}(a - a^\dagger)(S^- + S^+)$$

$$C_3 = \frac{\lambda\omega}{6\sqrt{N}}\left[\omega(a + a^\dagger)(S^+ + S^-) + 2\epsilon(a - a^\dagger)(S^- - S^+)\right.$$

$$\left.+ \frac{4\lambda}{\sqrt{N}}(S^- + S^+)^2\right]. \tag{18}$$

In order to evaluate the quality of the approximation we calculate the trace of C_3 that requires summation over both the atomic and the field variables. A sum over atomic variables is well-suited to calculating the trace, yielding

$$\sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} \langle S_1 \ldots S_N | C_3 | S_1 \ldots S_N \rangle = \frac{4}{3} \lambda^2 \omega. \quad (19)$$

The expression (8) is correct up to the third order in β (or rather for $\beta^3 \lambda^2 \omega < 1$), i.e. is an appropriate description for the high-temperature limit.

Although it is possible to derive higher-order approximations of partition function in a systematic manner, the increasing complexity of C_i for higher i now complicates the expressions in such way that $Z(N,T)$ is very difficult to evaluate analytically and requires numerical calculations [27]. It is important to note that also a good choice of decomposing the Hamiltonian may affect the complexity of the calculation and the error induced by the approximation (8). A different separation was studied in reference [20], in the simplest case of exact resonance between atom energy levels and frequency of radiation ($\omega = \epsilon$). The approximations that one obtains from this choice is derived splitting off from (14) the $\epsilon S^z / 2$ term too and is, for this reason, less accurate.

4 Partition function: Atomic variables

In the form of equation (8), the partition function may be performed analytically. Writing out explicitly the trace over the atomic variables, the partition function is given by

$$Z(N,T) = \text{Tr}_{\text{F}} \left\{ e^{-\beta \omega a^\dagger a} \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} \right.$$
$$\left. \times \langle S_1, \ldots, S_N | e^{-\beta \sum_{j=1}^N h_j} | S_1, \ldots, S_N \rangle \right\} \quad (20)$$

where

$$h_j = \frac{\epsilon}{2} \sigma_j^z + \frac{\lambda}{\sqrt{N}} (\sigma_j^+ + \sigma_j^-)(a^\dagger + a). \quad (21)$$

Noting that this operator has the property

$$[h_i, h_j] = 0, \quad (i \neq j) \quad (22)$$

from which it follows that

$$e^{-\beta \sum_{j=1}^N h_j} = \prod_{j=1}^N e^{-\beta h_j} \quad (23)$$

we can reduce the partition function to the simpler form

$$Z(N,T) = \text{Tr}_{\text{F}} \left\{ e^{-\beta \omega a^\dagger a} \right.$$
$$\left. \times \left[\sum_{S=\pm 1} \langle S | e^{-\beta \left[\frac{\epsilon}{2} \sigma^z + \frac{\lambda}{\sqrt{N}} (a^\dagger + a) \sigma^x \right]} | S \rangle \right]^N \right\} \quad (24)$$

where $\sigma^x \equiv \sigma^+ + \sigma^-$. Expanding the exponential operators in a power series, we obtain

$$e^{-\beta \left[\frac{\epsilon}{2} \sigma^z + \frac{\lambda}{\sqrt{N}} (a^\dagger + a) \sigma^x \right]} = \sum_{k=0}^\infty \frac{\beta^{2k}}{(2k)!} \left[\frac{\epsilon^2}{4} + \frac{\lambda^2}{N} (a^\dagger + a)^2 \right]^k$$
$$\times \left[I - \frac{\beta}{2k+1} \left(\frac{\epsilon}{2} \sigma^z + \frac{\lambda}{\sqrt{N}} (a^\dagger + a) \sigma^x \right) \right] \quad (25)$$

where we have used the following Pauli matrices properties

$$\sigma_z^2 = \sigma_x^2 = I, \quad \sigma^z \sigma^x + \sigma^x \sigma^z = 0. \quad (26)$$

Therefore, the sum of equation (24) is given by

$$\sum_{S=\pm 1} \langle S | e^{-\beta \left[\frac{\epsilon}{2} \sigma^z + \frac{\lambda}{\sqrt{N}} (a^\dagger + a) \sigma^x \right]} | S \rangle =$$
$$2 \sum_{k=0}^\infty \frac{\beta^{2k}}{(2k)!} \left[\frac{\epsilon^2}{4} + \frac{\lambda^2}{N} (a^\dagger + a)^2 \right]^k. \quad (27)$$

5 Partition function: Field variables

Using the Fock-state $|n\rangle$ for the photon field, the partition function is given by

$$Z(N,T) = 2^N \sum_{n=0}^\infty \exp(-\beta \omega n)$$
$$\times \langle n | \left\{ \sum_{k=0}^\infty \frac{\beta^{2k}}{(2k)!} \left[\frac{\epsilon^2}{4} + \frac{\lambda^2}{N} (a^\dagger + a)^2 \right]^k \right\}^N | n \rangle. \quad (28)$$

Since the operator $(a^\dagger + a)$ commutes with itself, the power series appearing into the above equation can be written

$$\left\{ \sum_{k=0}^\infty \frac{\beta^{2k}}{(2k)!} \left[\frac{\epsilon^2}{4} + \frac{\lambda^2}{N} (a^\dagger + a)^2 \right]^k \right\}^N = \left(\prod_{i=1}^N \sum_{k_i=0}^\infty \frac{\beta^{2k_i}}{(2k_i)!} \right)$$
$$\times \sum_{q=0}^K \frac{K!}{q!(K-q)!} \left(\frac{\epsilon}{2} \right)^{2(K-q)} \left(\frac{\lambda}{\sqrt{N}} \right)^{2q} (a^\dagger + a)^{2q} \quad (29)$$

where $K = k_1 + \cdots + k_N$. The matrix elements of photon operators are

$$\langle n | (a^\dagger + a)^{2q} | n \rangle = \frac{d^{2q}}{d\eta^{2q}} \langle n | e^{\eta(a^\dagger + a)} | n \rangle \Big|_{\eta=0}$$
$$= \frac{d^{2q}}{d\eta^{2q}} \left[e^{\frac{\eta^2}{2}} L_n(-\eta^2) \right] \Big|_{\eta=0} \quad (30)$$

where $L_n(x)$ is the nth Laguerre polynomial. At this point we are able to write down the partition function (28) as

$$Z(N,T) = 2^N \left(\prod_{i=1}^N \sum_{k_i=0}^\infty \frac{\beta^{2k_i}}{(2k_i)!} \right) \sum_{q=0}^K \frac{K!}{q!(K-q)!} \left(\frac{\epsilon}{2} \right)^{2(K-q)}$$
$$\times \left(\frac{\lambda}{\sqrt{N}} \right)^{2q} \frac{d^{2q}}{d\eta^{2q}} e^{\frac{\eta^2}{2}} \sum_{n=0}^\infty e^{-\beta \omega n} L_n(-\eta^2) \Big|_{\eta=0}. \quad (31)$$

The sum over n is well known [28] to be given by

$$\sum_{n=0}^{\infty} e^{-\beta\omega n} L_n(-\eta^2) = \frac{1}{1 - e^{-\beta\omega}} \exp\left[\eta^2 \frac{1}{e^{\beta\omega} - 1}\right]. \quad (32)$$

Substituting this into equation (31) one obtains

$$Z(N,T) = \frac{2^N}{1 - e^{-\beta\omega}} \left(\prod_{i=1}^{N} \sum_{k_i=0}^{\infty} \frac{\beta^{2k_i}}{(2k_i)!}\right) \sum_{q=0}^{K} \frac{K!}{q!(K-q)!}$$
$$\times \left(\frac{\epsilon}{2}\right)^{2(K-q)} \left(\frac{\lambda}{\sqrt{N}}\right)^{2q} \frac{d^{2q}}{d\eta^{2q}} \left. e^{\frac{\eta^2}{2} \coth \frac{\beta\omega}{2}}\right|_{\eta=0}. \quad (33)$$

At this stage we use the following result

$$\left.\frac{d^{2q}}{d\eta^{2q}} e^{\frac{\eta^2}{2} \coth \frac{\beta\omega}{2}}\right|_{\eta=0} = (2q-1)!! \coth^q \frac{\beta\omega}{2}, \quad q \geq 0 \quad (34)$$

and the integral representation for the double factorial [28]

$$(2q-1)!! = \frac{1}{2^{q+1}\sqrt{\pi}} \int_{-\infty}^{\infty} dz e^{-\frac{z^2}{4}} z^{2q}. \quad (35)$$

So, one has

$$Z(N,T) = \frac{1}{1 - e^{-\beta\omega}} \frac{2^N}{\sqrt{4\pi}} \int_{-\infty}^{\infty} dz e^{-\frac{z^2}{4}} \left(\prod_{i=1}^{N} \sum_{k_i=0}^{\infty} \frac{\beta^{2k_i}}{(2k_i)!}\right)$$
$$\times \sum_{q=0}^{K} \frac{K!}{q!(K-q)!} \left(\frac{\epsilon}{2}\right)^{2(K-q)} \left(\frac{\lambda z}{\sqrt{2N}}\right)^{2q} \coth^q \frac{\beta\omega}{2}$$
$$= \frac{1}{1 - e^{-\beta\omega}} \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} dz e^{-\frac{z^2}{4}}$$
$$\times \left\{ 2 \sum_{k=0}^{\infty} \frac{\beta^{2k}}{(2k)!} \left[\frac{\epsilon^2}{4} + \frac{\lambda^2 z^2}{2N} \coth \frac{\beta\omega}{2}\right]^k \right\}^N \quad (36)$$

which can be written in the final form

$$Z(N,T) = \frac{1}{1 - e^{-\beta\omega}} \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} dz e^{-\frac{z^2}{4}}$$
$$\times \left\{ 2 \cosh\left[\beta\sqrt{\frac{\epsilon^2}{4} + \frac{\lambda^2 z^2}{2N} \coth\left(\frac{\beta\omega}{2}\right)}\right] \right\}^N \quad (37)$$

where z is an order parameter. This result is independent from the choice of different states as a basis for expressing the state of radiation field and the same result can be derived using the coherent states $|\alpha\rangle$ for the photon field (see Appendix A).

6 Phase transition: classical limit

The integral (37) may be evaluated in the limit $N \to \infty$ by the steepest descent method. Writing the integral in term of a new variable $x = z/\sqrt{N}$ we search the value $\tilde{x}$ for which

$$f(x) = -\frac{x^2}{4} + \ln\left\{ 2 \cosh\left[\beta\sqrt{\frac{\epsilon^2}{4} + \frac{\lambda^2 x^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}\right] \right\} \quad (38)$$

is minimized. The minimum condition implies

$$\beta\lambda^2 \tanh\left[\beta\sqrt{\frac{\epsilon^2}{4} + \frac{\lambda^2 x^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}\right] =$$
$$\tanh\left(\frac{\beta\omega}{2}\right) \sqrt{\frac{\epsilon^2}{4} + \frac{\lambda^2 x^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}. \quad (39)$$

The existence of a nonzero solution of the above equation means a phase transition. Through equation (39) we can compute a critical temperature and we find

$$\beta_c = \frac{\epsilon}{2\lambda^2} \frac{\tanh(\beta_c \omega/2)}{\tanh(\beta_c \epsilon/2)}. \quad (40)$$

The quantity $\langle S_z \rangle$ is of physical interest. We calculate it as follows:

$$\langle S_z \rangle = -\frac{2}{\beta} \frac{\partial}{\partial\epsilon} \ln Z(N,T). \quad (41)$$

From equation (39) we infer that

$$\frac{\langle S_z \rangle}{N} = \begin{cases} -\tanh\left(\frac{\beta\epsilon}{2}\right), & \beta < \beta_c; \\ -\frac{\epsilon}{2\beta\lambda^2} \tanh\left(\frac{\beta\omega}{2}\right), & \beta > \beta_c. \end{cases} \quad (42)$$

A special case of interest is the limit reached when $\beta\omega \ll 1$. In this limit the partition function, which is given by equation (37), becomes

$$Z(N,T) \simeq \frac{1}{\sqrt{4\pi}\beta\omega} \int_{-\infty}^{\infty} dz e^{-\frac{z^2}{4}}$$
$$\times \left[2 \cosh\left(\beta\sqrt{\frac{\epsilon^2}{4} + \frac{\lambda^2 z^2}{N\beta\omega}}\right) \right]^N. \quad (43)$$

Let

$$y^2 = \frac{z^2}{4N\beta\omega} \quad (44)$$

one has

$$Z(N,T) \simeq \sqrt{\frac{N}{\beta\omega\pi}} \int_{-\infty}^{\infty} dy e^{-N\beta\omega y^2}$$
$$\times \left[2 \cosh\left(\beta\sqrt{\frac{\epsilon^2}{4} + 4\lambda^2 y^2}\right) \right]^N \quad (45)$$

that correspond to the partition function obtained with the Wang and Hioe computational method. In this limit, the square of the order parameter y represent the average number of photons. Just as for the Wang and Hioe

result [5], the critical temperature is obtained from the equation

$$\tanh\left(\frac{\beta_c \epsilon}{2}\right) = \frac{\epsilon \omega}{4\lambda^2} \tag{46}$$

and the model Hamiltonian (11) undergoes a phase transition at the critical value of the coupling constant $\lambda_c = \sqrt{\epsilon \omega}/2$. For a coupling $\lambda > \lambda_c$, above the critical temperature, the system is in the "normal phase", whereas for $\beta > \beta_c$, the equation (39) has a solution $\tilde{x} \neq 0$ and the system is in the so-called "super-radiant phase". In this region, the expectation value of the collective angular momentum operator (42) is

$$\frac{\langle S_z \rangle}{N} = \begin{cases} -\tanh\left(\frac{\beta\epsilon}{2}\right), & \beta < \beta_c; \\ -\frac{\epsilon\omega}{4\lambda^2}, & \beta > \beta_c. \end{cases} \tag{47}$$

7 Phase transition: Ising limit

The equivalence between the Dicke Model and the one-dimensional Ising Model for a system of mutually interacting spins $1/2$ embedded in a transverse magnetic field [29] has been studied with different techniques [13,14,20]. This similarity emerges in our approximate scheme when $\beta\epsilon \ll 1$, where equation (40) becomes

$$\tanh\left(\frac{\beta_c \omega}{2}\right) = \beta_c^2 \lambda^2. \tag{48}$$

This result may be easily obtained observing that, in this limit, our model is equivalent to a high-temperature expansion of the temperature-dependent effective Hamiltonian given by

$$H(\beta) = \omega\, a^\dagger a + \frac{\epsilon}{2} S_z - \frac{\beta\lambda^2}{2N}\coth\left(\frac{\beta\omega}{2}\right) S_x^2. \tag{49}$$

This effective Hamiltonian is the lowest-order effective Hamiltonian that can be obtained using the method of reference [14] (see Appendix B). As an alternative to the expression of Eq. (37), the partition function can be written as

$$Z(N,T) = \frac{1}{1 - e^{-\beta\omega}} \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} dz\, e^{-\frac{z^2}{4}}$$
$$\times \mathrm{Tr_A}\left\{ e^{-\beta\left[\frac{\epsilon}{2} S_z + \frac{\lambda z}{\sqrt{2N}}\sqrt{\coth\left(\frac{\beta\omega}{2}\right)} S_x\right]} \right\} \tag{50}$$

i.e., in the high temperature limit,

$$Z(N,T) \simeq \frac{1}{1 - e^{-\beta\omega}} \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} dz\, e^{-\frac{z^2}{4}}$$
$$\times \mathrm{Tr_A}\left\{ e^{-\beta\frac{\epsilon}{2} S^z} e^{\frac{\beta\lambda z}{\sqrt{2N}}\sqrt{\coth\left(\frac{\beta\omega}{2}\right)} S_x} \right\}$$
$$= \frac{1}{1 - e^{-\beta\omega}} \mathrm{Tr_A}\left\{ e^{-\beta\frac{\epsilon}{2} S^z} e^{\frac{\beta^2\lambda^2}{2N}\coth\left(\frac{\beta\omega}{2}\right) S_x^2} \right\}. \tag{51}$$

As in the most general case (37), we can write the integral in term of $x = z/\sqrt{N}$ and utilize (in the limit $N \to \infty$) the steepest descent method in order to search the value $\tilde{x}$ for which

$$f(x) = -\frac{x^2}{4} + \ln\left\{ 2\cosh\left(\frac{\beta\epsilon}{2}\right) \right.$$
$$\left. \times \cosh\left[\frac{\beta\lambda z}{\sqrt{2N}}\sqrt{\coth\left(\frac{\beta\omega}{2}\right)}\right] \right\} \tag{52}$$

is minimized. The minimum condition implies

$$\frac{\tilde{x}}{2} = \frac{\beta\lambda}{\sqrt{2}}\sqrt{\coth\left(\frac{\beta\omega}{2}\right)}\tanh\left[\frac{\beta\lambda\tilde{x}}{\sqrt{2}}\sqrt{\coth\left(\frac{\beta\omega}{2}\right)}\right]. \tag{53}$$

The existence of a nonzero solution of the above equation means a phase transition. Through equation (53) we can compute a critical temperature that we find to be identical to the result (48). The order parameter near the critical temperature is

$$\tilde{x} \simeq 2\sqrt{3}\,\sqrt{\frac{\beta - \beta_c}{\beta_c}}. \tag{54}$$

In the limit $\beta_c\omega \ll 1$, we get $\beta_c \simeq \omega/2\lambda^2$ and the interaction of independent spins with a photon mode induce an effective spin-spin interaction of long range nature that may be described by the Hamiltonian

$$H = \omega\, a^\dagger a - \frac{\lambda^2}{N\omega} S_x^2 \tag{55}$$

that is in agreement with the results of references [10,13].

Before leaving this section, we note that, for the special case of exact resonance $\omega = \epsilon$, equation (40) reduces to

$$\beta_c = \frac{\epsilon}{2\lambda^2} \tag{56}$$

and this lead to the absence of a critical value of the coupling constant, i.e. to the result that the phase transition could occur even for $\lambda < \epsilon/2$. However, the approximation that we have proposed is valid for $\beta_c^3\lambda^2\omega < 1$ and this condition permit us to derive accurate results only for $\lambda > \epsilon/2$, i.e. for coupling sufficiently higher that λ_c.

8 Conclusion

The thermodynamic properties of a system governed by the Dicke Hamiltonian have been treated in the framework of an approximate model, through a perturbative expansion of partition function obtained by decomposing the Hamiltonian into two non-commuting hermitian operators. This technique represent a practical and convenient method to determinate the behavior of the Dicke model in the strong-coupling regime. We have obtained a simple analytic expression for the partition function and the critical temperature is easily determined. This technique was then

used to explore some limiting cases and old and recent results are derived in an elementary and unified way. Our result extend those of reference [20] beyond the resonant interaction condition and support the recent arguments on the subject [13,14] concerning the similarity between the Dicke and collective one-dimensional Ising model.

Appendix A: Coherent states

In terms of coherent states $|\alpha\rangle$ [18], the partition function is given by

$$
Z(N,T) = 2^N \int \frac{d^2\alpha}{\pi} \langle\alpha|e^{-\beta\omega a^\dagger a}
$$
$$
\times \left\{ \sum_{k=0}^{\infty} \frac{\beta^{2k}}{(2k)!} \left[\frac{\epsilon^2}{4} + \frac{\lambda^2}{N}(a^\dagger + a)^2 \right]^k \right\}^N |\alpha\rangle \quad (57)
$$

where $a|\alpha\rangle = \alpha|\alpha\rangle$ and

$$
\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1. \quad (58)
$$

Using the over-completeness of the field coherent states (58) and the result

$$
\langle\alpha|e^{-\beta\omega a^\dagger a}|\gamma\rangle = \langle\alpha|\gamma\rangle e^{-\alpha^*\gamma[1-\exp(-\beta\omega)]}. \quad (59)
$$

Equation (57) takes the form

$$
Z(N,T) = 2^N \int \frac{d^2\alpha}{\pi} \int \frac{d^2\gamma}{\pi} \langle\alpha|\gamma\rangle e^{-\alpha^*\gamma[1-\exp(-\beta\omega)]}
$$
$$
\times \langle\gamma| \left\{ \sum_{k=0}^{\infty} \frac{\beta^{2k}}{(2k)!} \left[\frac{\epsilon^2}{4} + \frac{\lambda^2}{N}(a^\dagger + a)^2 \right]^k \right\}^N |\alpha\rangle. \quad (60)
$$

The partition function may be derived from equation (29) by computing the matrix element

$$
\langle\gamma|(a^\dagger + a)^{2q}|\alpha\rangle = \frac{d^{2q}}{d\eta^{2q}} \langle\gamma|e^{\eta(a^\dagger + a)}|\alpha\rangle \bigg|_{\eta=0}
$$
$$
= \langle\gamma|\alpha\rangle \frac{d^{2q}}{d\eta^{2q}} e^{\frac{\eta^2}{2}} e^{\eta(\gamma^*+\alpha)} \bigg|_{\eta=0} \quad (61)
$$

and by using

$$
|\langle\alpha|\gamma\rangle|^2 = e^{-|\alpha-\gamma|^2}. \quad (62)
$$

One obtains

$$
Z(N,T) = 2^N \left(\prod_{i=1}^{N} \sum_{k_i=0}^{\infty} \frac{\beta^{2k_i}}{(2k_i)!} \right)
$$
$$
\times \sum_{q=0}^{K} \frac{K!}{q!(K-q)!} \left(\frac{\epsilon}{2} \right)^{2(K-q)} \left(\frac{\lambda}{\sqrt{N}} \right)^{2q} \frac{d^{2q}}{d\eta^{2q}} e^{\frac{\eta^2}{2}}
$$
$$
\times \int \frac{d^2\alpha}{\pi} \int \frac{d^2\gamma}{\pi} e^{-|\alpha|^2-|\gamma|^2+\gamma^*\alpha+\alpha^*\gamma e^{-\beta\omega}+\eta(\gamma^*+\alpha)} \bigg|_{\eta=0}. \quad (63)
$$

Recalling that the integration measure is defined to be given by

$$
\frac{d^2\alpha}{\pi} = \frac{d\alpha d\alpha^*}{2i} = \frac{d(\mathrm{Re}\alpha)\, d(\mathrm{Im}\alpha)}{\pi} \quad (64)
$$

one finds the partition function expression of equation (33), i.e. the same result obtained with the use of Fock states as a basis for the photon field. Finally, we want to underline that our results are derived without the c-number substitution for the field variables $a \to \alpha$, the Wang and Hioe computational method, that permits to deal with non-interacting atoms subjected to an external magnetic field described by the amplitude α.

Appendix B: Effective Hamiltonian

In this Appendix we will discuss the approach to the problem of constructing effective atomic Hamiltonian for the Dicke Model. The Zassenhaus formula (3) can be used to obtain the following result:

$$
e^{-\beta H_I} \prod_{i=2}^{\infty} e^{(-\beta)^i C_i} = \sum_{i=0}^{\infty} \frac{(-\beta)^i}{i!} P_i \quad (65)
$$

where

$$
P_i = H_I^i + \sum_{k=2}^{i} \frac{i!}{(i-k)!} H_I^{i-k} C_k. \quad (66)
$$

The partition function of the whole system can be written as [27]:

$$
Z(N,T) = \mathrm{Tr}_A \left[\mathrm{Tr}_F \left(e^{-\beta H_0} \sum_{i=0}^{\infty} \frac{(-\beta)^i}{i!} P_i \right) \right]
$$
$$
= \mathrm{Tr}_F \left(e^{-\beta H_0} \right) \mathrm{Tr}_A \left(e^{-\beta H_A^{eff}} \right) \quad (67)
$$

where

$$
H_A^{eff} = -\frac{1}{\beta} \ln \left\langle \sum_{i=0}^{\infty} \frac{(-\beta)^i}{i!} P_i \right\rangle_F
$$
$$
= \frac{1}{\beta} \sum_{q=1}^{\infty} \frac{(-1)^q}{q} \left(\sum_{i=1}^{\infty} \frac{(-\beta)^i}{i!} \langle P_i \rangle_F \right)^q \quad (68)
$$

where the thermal averaging is carried out with respect to the field variables, i.e. $\langle O \rangle_F = \mathrm{Tr}_F \left(e^{-\beta H_0} O \right) / \mathrm{Tr}_F \left(e^{-\beta H_0} \right)$. Eq. (68) can be rewritten as

$$
H_A^{eff} = \sum_{q=1}^{\infty} \frac{\beta^{q-1}}{q!} Q_q \quad (69)
$$

where the lowest-order terms are

$$
Q_1 = \langle P_1 \rangle_F \equiv \langle H_I \rangle_F = \frac{\epsilon}{2} S_z,
$$
$$
Q_2 = \langle P_1 \rangle_F^2 - \langle P_2 \rangle_F \equiv \langle H_I \rangle_F^2 - \langle H_I^2 \rangle_F
$$
$$
= -\frac{\lambda^2}{N} \coth \left(\frac{\beta\omega}{2} \right) S_x^2. \quad (70)
$$

Therefore the lowest-order effective Hamiltonian is

$$H_A^{eff} = \frac{\epsilon}{2}S_z - \frac{\beta\lambda^2}{2N}\coth\left(\frac{\beta\omega}{2}\right)S_x^2 \qquad (71)$$

in agreement with the result of reference [14].

Appendix C: Inclusion of diamagnetic term

In this Appendix we briefly discuss the inclusion of diamagnetic effects in our model. Following the discussion of references [22] and [20], the diamagnetic term may be incorporated in the Dicke Hamiltonian of equation (11), without adding significant complication of the problem, by using a new field-mode of frequency Ω instead of frequency ω, which is given by

$$\Omega = \sqrt{\omega(\omega + 4k)} \qquad (72)$$

where $\omega k = e^2\pi\rho/m$. In terms of this new frequency, the model Hamiltonian becomes

$$H = \Omega a^\dagger a + \sum_{i=1}^{N}\left[\frac{\epsilon}{2}\sigma_i^z + \frac{\Lambda}{\sqrt{N}}(a^\dagger + a)(\sigma_i^+ + \sigma_i^-)\right] \qquad (73)$$

where

$$\Lambda = \epsilon d\sqrt{\frac{2\pi\rho}{\Omega}}. \qquad (74)$$

The Hamiltonian (73) is formally identical to the Hamiltonian (11) and can be used to derive thermodynamic results. The phase transition that is obtained in the limit of high coupling constant ($\Lambda \gg \Lambda_c = \sqrt{\epsilon\Omega}/2$) cannot occur due to sum-rule arguments [21] that requires $\Lambda < \sqrt{\epsilon\Omega}/2$.

References

1. R.H. Dicke, Phys. Rev. **93**, 99 (1954)
2. K. Hepp, A. Lieb, Ann. Phys. **76**, 360 (1973)
3. Y.K. Wang, F.T. Hioe, Phys. Rev. A **7**, 831 (1973); F.T. Hioe, Phys. Rev. A **8**, 1440 (1973)
4. G. Vertogen, A.S. De Vries, Phys. Lett. A **48**, 451 (1974)
5. G.C. Duncan, Phys. Rev. A **9**, 418 (1974)
6. R. Gilmore, C.M. Bowden, Phys. Rev. A **13**, 1898 (1976)
7. M. Yamanoi, J. Phys. A: Math. Gen. **9**, 573 (1976)
8. M. Orszag, J. Phys. A: Math. Gen. **10**, L21 (1977); M. Orszag, J. Phys. A: Math. Gen. **10**, 1995 (1977)
9. S. Sivasubramanian, A. Widom, Y. Shrivastava, Physica A **301**, 241 (2001)
10. C. Emary, T. Brandes, Phys. Rev. Lett. **90**, 044101 (2003)
11. C. Emary, T. Brandes, Phys. Rev. E **67**, 66203 (2003)
12. X. Hou, B. Hu, Phys. Rev. A **69**, 042110 (2004)
13. N. Lambert, C. Emary, T. Brandes, Phys. Rev. Lett. **92**, 073602 (2004)
14. J. Reslen, L. Quiroga, N.F. Johnson, Europhys. Lett. **69**, 8 (2005)
15. S. John, T. Quang, Phys. Rev. Lett. **76**, 1320 (1996)
16. C.F. Lee, N.F. Johnson, Phys. Rev. Lett. **93**, 83001 (2004)
17. M. Frasca, cond-mat/0411171
18. R.J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963)
19. C.F. Lee, N.F. Johnson, quant-ph/0412069
20. G. Liberti, R.L. Zaffino, Phys. Rev. A **70**, 033808 (2004)
21. K. Rzazewski, K. Wodkiewicz, W. Zakowicz, Phys. Rev. Lett. **35**, 432 (1975); I. Bialynicki-Birula, K. Rzazewski, Phys. Rev. A **19**, 301 (1979); K. Rzazewski, J. Phys. A: Math. Gen. **12**, 2115 (1979); K. Gawedzki, K. Rzazewski, Phys. Rev. A **23**, 2134 (1981)
22. M.D. Crisp, Phys. Rev. A **44**, 563 (1991); M.D. Crisp, Phys. Rev. A **46**, 4138 (1992)
23. H. De Raedt, B. De Raedt, Phys. Rev. A **28**, 3575 (1983)
24. M. Vanhimbeeck, H. De Raedt, D. Schoemeker, Phys. Rev. A **36**, 4612 (1987)
25. G. Polatsek, K.W. Becker, Phys. Rev. B **55**, 16096 (1997)
26. R.M. Wilcox, J. Math. Phys. **8**, 962 (1967)
27. P. de Vries, A. Lagendijk, H. De Raedt, Phys. Rev. B **35**, 5425 (1987)
28. G. Arfken, *Mathematical Methods for Physicists* (Academic Press Inc., San Diego, 1985)
29. G. Parisi, *Statistical Field Theory* (Addison-Wesley P.C., Redwood, 1988)